Applied Mathematics Letters 24 (2011) 627–629

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

The Menger number of the Cartesian product of graphs[☆]

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ARTICLE INFO

Article history:
Received 31 August 2010
Accepted 24 November 2010

Keywords:
Interconnection network
Menger number
Cartesian product
Path

ABSTRACT

In a real-time system, the Menger number $\zeta_l(G)$ is an important measure of the communication efficiency and fault tolerance of the system G. In this paper, we obtain a lower bound for the Cartesian product graph. We show that $\zeta_{l_1+l_2}(G_1\times G_2)\geq \zeta_{l_1}(G_1)+\zeta_{l_2}(G_2)$ for $l_1\geq 2$ and $l_2\geq 2$. As an application of the result, we determine the exact values $\zeta_l(G)$ of the grid network $G=G(m_1,m_2,\ldots,m_n)$ for $m_i\geq 2$ $(1\leq i\leq n)$ and $l\geq\sum_{i=1}^n m_i$. This example shows that the lower bound of $\zeta_{l_1+l_2}(G_1\times G_2)$ obtained is tight.

1. Introduction

It is well known that underlying topology of an interconnection network can be represented by a graph $G=(V,E)$, where V is the set of processors and E is the set of communication links in the network. Throughout this paper, a graph $G=(V,E)$ always means a connected and simple graph (without loops and multiple edges), where $V=V(G)$ and $E=E(G)$ are the vertex set and the edge set of G, respectively. For graph terminology and notation not defined here, we follow [1].

Let x and y be two distinct vertices in a graph $G=(V,E)$. A path between x and y is denoted by the term xy-path. The distance $d_G(x,y)$ between x and y is the number of edges in a shortest xy-path, and the diameter of G is $d(G)=\max\{d_G(x,y): x,y\in V(G)\}$. For a vertex $x\in V(G)$, the set of neighbors of x is denoted by $N_G(x)$ in G and the degree of x is $d_G(x)=|N_G(x)|$. The minimum degree of G is $\delta(G)=\min\{d_G(v): v\in V(G)\}$.

When we use a graph to model a parallel computing or processing system, we can use internally disjoint paths to transmit messages simultaneously from a vertex x to another vertex y. However, in a real-time system, the message delay must be limited within a given period since any message obtained beyond the bound may be worthless. A natural question is how many internally disjoint paths exist in the network to ensure message delay within the effective bounds. In the language of graph theory, this problem can be stated as follows.

Let x and y be two distinct vertices in a graph G. The xy-Menger number with respect to l, denoted by $\zeta_l(x,y)$, is the maximum number of internally disjoint xy-paths whose lengths are at most l in G. The Menger number of G with respect to l is defined as $\zeta_l(G)=\min\{\zeta_l(x,y): x,y\in V(G)\}$. If $l<d(G)$, then $\zeta_l(G)=0$. To avoid the relatively trivial case in which $l<d(G)$ or G is a complete graph, we assume that $l\geq d(G)\geq 2$ in this paper. Clearly, $\zeta_l(G)\leq\delta(G)$. For a graph G with $d(G)\geq 2$ and $|V(G)|=n$, it is clear that $\zeta_l(G)$ is well defined for an integer l with $d(G)\leq l\leq n-1$ and $\zeta_{d(G)}(G)\leq\zeta_{d(G)+1}(G)\leq\cdots\leq\zeta_{n-1}(G)$. There are many papers that have studied Menger-type parameters, such as [2–8].

We consider the Cartesian product $G_1\times G_2$ of graphs G_1 and G_2. For graphs G_1 and G_2, the Cartesian product $G_1\times G_2$ is the graph with vertex set $V(G_1\times G_2)=V(G_1)\times V(G_2)=\{xy\mid x\in V(G_1), y\in V(G_2)\}$ and edge set $E(G_1\times G_2)=\{(x_1x_2,y_1y_2)\mid x_1=y_1 \text{ and } (x_2,y_2)\in E(G_2) \text{ or } x_2=y_2 \text{ and } (x_1,y_1)\in E(G_1)\}$. It is well known that the Cartesian product is an important

☆ This work is partially supported by NSFC (No. 10971198) and Zhejiang Innovation Project (No. T200905).
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doi:10.1016/j.aml.2010.11.026

research topic in graph theory (see, e.g., [9–13]). It is also well known that, for designing large-scale interconnection networks, the Cartesian product is an important method to obtain large graphs from smaller ones, with a number of parameters that can be easily calculated from the corresponding parameters for those small initial graphs. The Cartesian product preserves many nice properties such as regularity, existence of Hamilton cycles and Euler circuits, and transitivity of the initial graphs (see, e.g., [1]). In fact, many well-known networks can be constructed by the Cartesian products of some simple graphs. For example, a torus is the Cartesian product of two cycles, a mesh is the Cartesian product of two paths, and a grid is the Cartesian product of several paths. What we are interested in is the Menger number of the Cartesian product of graphs.

2. Main results

For a vertex $x \in V(G_1)$ and a subgraph $H \subseteq G_2$, we use xH to denote the subgraph of $G_1 \times G_2$ induced by $\{x\} \times V(H)$. Similarly, for a vertex $y \in V(G_2)$, and a subgraph $H \subseteq G_2$, Hy denotes the subgraph of $G_1 \times G_2$ induced by $V(H) \times \{y\}$. The symbol $l(P)$ denotes the length of a path P, which is the number of edges in P.

Now, we state our main result of this paper.

Theorem 1. *For any two connected graphs G_1 and G_2, if $l_i \geq 2$ for $i = 1, 2$, then $\zeta_{l_1+l_2}(G_1 \times G_2) \geq \zeta_{l_1}(G_1) + \zeta_{l_2}(G_2)$.*

Proof. Assume that $x = x_1x_2$ and $y = y_1y_2$ are two distinct vertices in $G_1 \times G_2$, where $x_1, y_1 \in V(G_1)$ and $x_2, y_2 \in V(G_2)$.

If $x_1 \neq y_1$, there must exist $\zeta_{l_1}(G_1)$ internally disjoint x_1y_1-paths $P_1, P_2, \ldots, P_{\zeta_{l_1}(G_1)}$ in G_1 such that $l(P_i) \leq l_1$ for any $i \in \{1, 2, \ldots, \zeta_{l_1}(G_1)\}$. Without loss of generality, we may assume that $l(P_1) \leq l(P_2) \leq \cdots \leq l(P_{\zeta_{l_1}(G_1)})$. Then $l(P_i) \geq 2$ for any $i \in \{2, \ldots, \zeta_{l_1}(G_1)\}$. Let v_i be the first internal vertex in P_i $(2 \leq i \leq \zeta_{l_1}(G_1))$. It is clear that $v_i \in N_{G_1}(x_1)$. Then v_i splits the path P_i into two subpaths a_i and P_i', where a_i is the first edge (x_1, v_i) in P_i and P_i' is the subpath of P_i from v_i to y_1. Hence the path P_i can be expressed as

$$P_i = x_1 \xrightarrow{a_i} v_i \xrightarrow{P_i'} y_1, \quad i = 2, 3, \ldots, \zeta_{l_1}(G_1).$$

Similarly, if $x_2 \neq y_2$, there must exist $\zeta_{l_2}(G_2)$ internally disjoint x_2y_2-paths $W_1, W_2, \ldots, W_{\zeta_{l_2}(G_2)}$ in G_2 such that $l(P_j) \leq l_2$ for any $j \in \{1, 2, \ldots, \zeta_{l_2}(G_2)\}$. Without loss of generality, we may assume that $l(W_1) \leq l(W_2) \leq \cdots \leq l(W_{\zeta_{l_2}(G_2)})$. Then $l(W_j) \geq 2$ for any $j \in \{2, \ldots, \zeta_{l_2}(G_2)\}$. Let u_j be the first internal vertex in P_j $(2 \leq j \leq \zeta_{l_2}(G_2))$. Then the path W_j can be expressed as $W_j = x_2 \xrightarrow{b_j} u_j \xrightarrow{W_j'} y_2, j = 2, 3, \ldots, \zeta_{l_2}(G_2)$, where b_j is the first edge (x_2, u_j) in W_j and W_j' is the subpath of W_j from u_j to y_2. It is clear that $u_j \in N_{G_2}(x_2)$.

Using the above notations, we can construct $\zeta_{l_1}(G_1) + \zeta_{l_2}(G_2)$ internally disjoint xy-paths $R_1, R_2, \ldots R_{\zeta_{l_1}(G_1)+\zeta_{l_2}(G_2)}$ with $l(R_i) \leq l_1 + l_2$ for each i. Consider the following three cases.

Case 1. $x_1 \neq y_1, x_2 \neq y_2$.

Let $R_1 = x_1x_2 \xrightarrow{P_1x_2} y_1x_2 \xrightarrow{y_1W_1} y_1y_2$; then $l(R_1) = l(P_1) + l(W_1) \leq l_1 + l_2$.

For $i = 2, 3, \ldots, \zeta_{l_1}(G_1)$, let $R_i = x_1x_2 \xrightarrow{a_ix_2} v_ix_2 \xrightarrow{v_iW_1} v_iy_2 \xrightarrow{P_i'y_2} y_1y_2$; then $l(R_i) = 1 + l(W_1) + l(P_i') \leq l_1 + l_2$.

Let $R_{\zeta_{l_1}(G_1)+1} = x_1x_2 \xrightarrow{x_1W_1} x_1y_2 \xrightarrow{P_1y_2} y_1y_2$; then $l(R_1) = l(W_1) + l(P_1) \leq l_1 + l_2$.

For $j = 2, 3, \ldots, \zeta_{l_2}(G_2)$, let $R_{\zeta_{l_1}(G_1)+j} = x_1x_2 \xrightarrow{x_1b_j} x_1u_j \xrightarrow{P_1u_j} y_1u_j \xrightarrow{y_1W_j'} y_1y_2$; then $l(R_{\zeta_{l_1}(G_1)+j}) = 1+l(P_1)+l(W_j') \leq l_1+l_2$.

Case 2. $x_1 = y_1, x_2 \neq y_2$.

Since $|N_{G_1}(x_1)| = d_{G_1}(x_1) \geq \delta(G_1) \geq \zeta_{l_1}(G_1)$, $N_{G_1}(x_1) \setminus \{v_2, v_3, \ldots, v_{\zeta_{l_1}(G_1)}\} \neq \emptyset$. Let $v_1 \in N_{G_1}(x_1) \setminus \{v_2, v_3, \ldots, v_{\zeta_{l_1}(G_1)}\}$ and $a_1 = (x_1, v_1)$.

For $i = 1, 2, \ldots, \zeta_{l_1}(G_1)$, let $R_i = x_1x_2 \xrightarrow{a_ix_2} v_ix_2 \xrightarrow{v_iW_1} v_iy_2 \xrightarrow{a_iy_2} y_1y_2$; then $l(R_i) = 1 + l(W_1) + 1 \leq l_1 + l_2$.

For $j = 1, 2, \ldots, \zeta_{l_2}(G_2)$, let $R_{\zeta_{l_1}(G_1)+j} = x_1x_2 \xrightarrow{x_1W_j} x_1y_2 = y_1y_2$; then $l(R_{\zeta_{l_1}(G_1)+j}) = l(W_j) < l_1 + l_2$.

Case 3. $x_1 \neq y_1, x_2 = y_2$.

For $i = 1, 2, \ldots, \zeta_{l_1}(G_1)$, let $R_i = x_1x_2 \xrightarrow{P_ix_2} y_1x_2 = y_1y_2$; then $l(R_i) = l(P_i) < l_1 + l_2$.

Since $|N_{G_2}(x_2)| = d_{G_2}(x_2) \geq \delta(G_2) \geq \zeta_{l_2}(G_2)$, $N_{G_2}(x_2) \setminus \{u_2, u_3, \ldots, u_{\zeta_{l_2}(G_2)}\} \neq \emptyset$. Let $u_1 \in N_{G_2}(x_2) \setminus \{u_2, u_3, \ldots, u_{\zeta_{l_1}(G_1)}\}$ and $b_1 = (x_2, u_1)$.

For $j = 1, 2, \ldots, \zeta_{l_2}(G_2)$, let $R_{\zeta_{l_1}(G_1)+j} = x_1x_2 \xrightarrow{x_1b_j} x_1u_j \xrightarrow{P_1u_j} y_1u_j \xrightarrow{y_1b_j} y_1x_2 = y_1y_2$; then $l(R_{\zeta_{l_1}(G_1)+j}) = 1 + l(W_j) + 1 \leq l_1 + l_2$.

It is easy to check that the xy-paths $R_1, R_2, \ldots, R_{\zeta_{l_1}(G_1)+\zeta_{l_2}(G_2)}$ constructed above are internally disjoint in $G_1 \times G_2$ whichever case occurs.

Since $l(R_i) \leq l_1 + l_2$ for $1 \leq i \leq \zeta_{l_1}(G_1) + \zeta_{l_2}(G_2)$, the theorem follows. $\square$

The connectivity of a graph G, denoted by $\kappa(G)$, is the minimum number of vertices whose removal leaves the remaining graph disconnected or trivial. It follows from Menger's theorem that $\kappa(G) \geq k$ if any two distinct vertices of G are connected by at least k internal vertex-disjoint paths. Generally, we have $\zeta_l(G) \leq \kappa(G)$. If P is a path of length m, then $\zeta_l(P) = 1 = \kappa(P)$ for any $l \geq m$.

The grid network is defined as $G(m_1, m_2, \ldots, m_n) = P_{m_1} \times P_{m_2} \times \cdots \times P_{m_n}$, where P_{m_i} is a path of length m_i for each $i = 1, 2, \ldots, n$. As an application of Theorem 1, we obtain the Menger number of the grid network. The following lemma is useful in the proof of our conclusion.

Lemma 2 (*Theorems 2.3.3 and 2.3.4 in [1]*). *Let $G_1, G_2, \ldots, G_n$ be n simple graphs. Then $d(G_1 \times G_2 \times \cdots \times G_n) = d(G_1) + d(G_2) + \cdots + d(G_n)$. If $\kappa(G_i) = \delta(G_i) > 0$ for each $i = 1, 2, \ldots, n$, then $\kappa(G_1 \times G_2 \times \cdots \times G_n) = \kappa(G_1) + \kappa(G_2) + \cdots + \kappa(G_n)$.*

Corollary 3. *Let $G = G(m_1, m_2, \ldots, m_n)$ be a grid network. If $l \geq \sum_{i=1}^{n} m_i$ and $m_i \geq 2$ for each $i = 1, 2, \ldots, n$, then $\zeta_l(G) = \zeta_{m_1}(P_{m_1}) + \zeta_{m_2}(P_{m_2}) + \cdots + \zeta_{m_n}(P_{m_n}) = n$.*

Proof. Since $d(P_{m_i}) = m_i$, by Lemma 2, we have $d(G) = \sum_{i=1}^{n} m_i$. For $l \geq d(G)$, we have $\zeta_l(G) \geq \zeta_{d(G)}(G)$. By Theorem 1, using the associative law, we have $\zeta_{d(G)}(G) \geq \zeta_{m_1}(P_{m_1}) + \zeta_{m_2}(P_{m_2}) + \cdots + \zeta_{m_n}(P_{m_n}) = n$. By Lemma 2, we have $\kappa(G) = n$. By $\zeta_l(G) \leq \kappa(G) = n$ and $\zeta_l(G) \geq \zeta_{d(G)}(G) = n$, we have $\zeta_l(G) = n = \zeta_{m_1}(P_{m_1}) + \zeta_{m_2}(P_{m_2}) + \cdots + \zeta_{m_n}(P_{m_n})$.

The corollary is proved. □

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